Relativistic Effects in Femtoscopy & Deuteron Formation

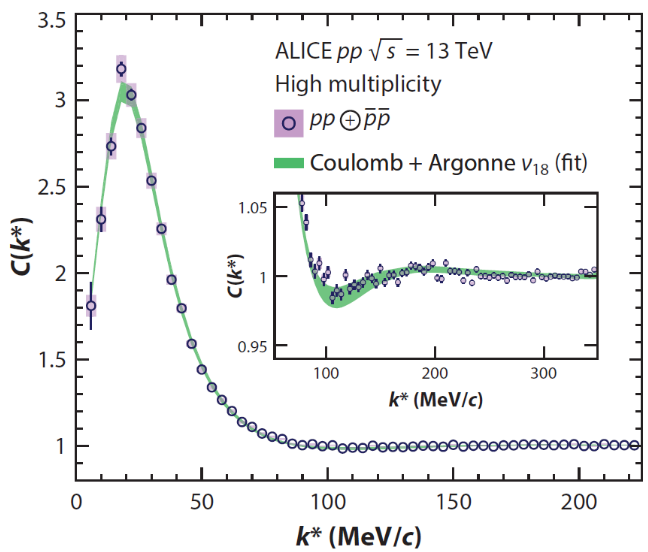
Stanisław Mrówczyński

National Centre for Nuclear Research, Warsaw, Poland

Why relativistic effects need to be discussed?

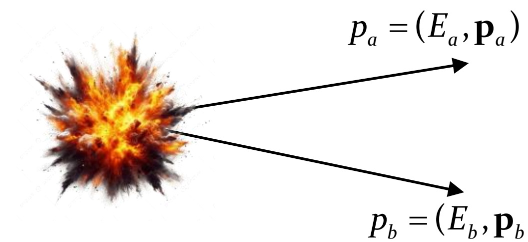
- Measurements of two particle correlations are usually performed in the CM frame where the particles are non-relativistic. The motion of the pair with respect to the particles' source is truly relativistic.

- Femtoscopic measurements are very precise nowadays. Are theoretical predictions equally accurate?



L. Fabbietti, V. Mantovani Sarti & O.Vazquez Doce, Ann. Rev. Nucl. Part. Sci. **71**, 377 (2021)

Correlation function



Definition

$$\boxed{\frac{dN_{ab}}{d\mathbf{p}_a d\mathbf{p}_b} = C(\mathbf{p}_a, \mathbf{p}_b) \frac{dN_a}{d\mathbf{p}_a} \frac{dN_b}{d\mathbf{p}_b}}$$

Correlation function
is a Lorentz scalar

$$\underbrace{E_a E_b \frac{dN_{ab}}{d\mathbf{p}_a d\mathbf{p}_b}}_{\text{Lorentz invariant}} = C(p_a, p_b) \underbrace{E_a \frac{dN_a}{d\mathbf{p}_a} E_b \frac{dN_b}{d\mathbf{p}_b}}_{\text{Lorentz invariant}}$$

Lorentz transformation $\quad p \equiv (E, \mathbf{p}) \to p' \equiv (E', \mathbf{p}') = Lp$

$$\boxed{C(p_a, p_b) \to C'(p'_a, p'_b) = C(L^{-1} p'_a, L^{-1} p'_b)}$$

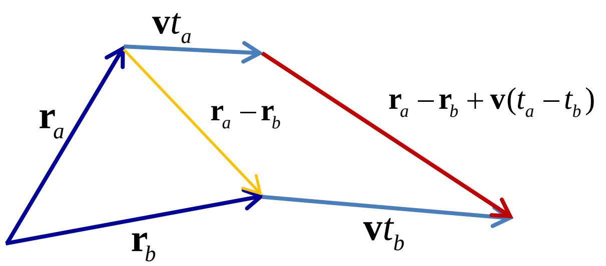
Non-relativistic Koonin formula

wave function

$$\boxed{C(\mathbf{p}_a,\mathbf{p}_b) = \int dt_a d^3 r_a \int dt_b d^3 r_b \, S^a(\mathbf{r}_a) S^b(\mathbf{r}_b) \left|\psi(\mathbf{r}_a + \mathbf{v}t_a, \mathbf{r}_b + \mathbf{v}t_b)\right|^2}$$

source functions

$$\int dt\, d^3\mathbf{r}\, S^{a,b}(t,\mathbf{r}) = 1$$



CM variables

$$\begin{cases} \mathbf{R} \equiv \dfrac{m_a \mathbf{r}_a + m_b \mathbf{r}_b}{m_a + m_b}, & \mathbf{r} \equiv \mathbf{r}_a - \mathbf{r}_b \\ T \equiv \dfrac{m_a t_a + m_b t_b}{m_a + m_b}, & t \equiv t_a - t_b \\ \mathbf{P} \equiv \mathbf{p}_a + \mathbf{p}_b, & \mathbf{q} \equiv \dfrac{m_b \mathbf{p}_a + m_a \mathbf{p}_b}{m_a + m_b} \end{cases}$$

$$S_r(t,\mathbf{r}) \equiv \int dT\, d^3\mathbf{R}\, S^a\left(T - \frac{1}{2}t, \mathbf{R} - \frac{1}{2}\mathbf{r}\right) S^b\left(T + \frac{1}{2}t, \mathbf{R} + \frac{1}{2}\mathbf{r}\right) \qquad \psi(\mathbf{r}_a,\mathbf{r}_b) = e^{i\mathbf{P}\cdot\mathbf{R}}\varphi_{\mathbf{q}}(\mathbf{r})$$

$$S_r^{eff}(\mathbf{r}) \equiv \int dt\;\; S_r\left(t, \mathbf{r} - \mathbf{v}t\right)$$

$$\int d^3\mathbf{r}\, S_r^{eff}(\mathbf{r}) = 1$$

$$\boxed{C(\mathbf{P},\mathbf{q}) = \int d^3 r\, S_r^{\mathrm{eff}}(\mathbf{r}) \left|\varphi_{\mathbf{q}}(\mathbf{r})\right|^2}$$

The formula holds in the CM frame
of correlated particles!

S.E. Koonin, Phys. Lett. B **70**, 43 (1977)

Relativistically covariant source function

Emission probability is Lorentz invarinat $S(x)d^4x = S'(x')d^4x' \qquad x \equiv (t, \mathbf{r})$

Since $d^4x = d^4x'$ the source function is a Lorentz scalar $S(x) \to S'(x') = S(L^{-1}x')$

Gaussian source function

$$S(x) = \frac{\sqrt{\det \Lambda}}{4\pi^2} \exp\left[-\frac{1}{2} x_\mu \Lambda^{\mu\nu} x_\nu \right]$$

In source rest frame

$$\Lambda^{\mu\nu} = \begin{bmatrix} \tau^{-2} & 0 & 0 & 0 \\ 0 & R_x^{-2} & 0 & 0 \\ 0 & 0 & R_y^{-2} & 0 \\ 0 & 0 & 0 & R_z^{-2} \end{bmatrix}$$

$$S(x) = \frac{1}{4\pi^2 \tau R_x R_y R_z} \exp\left[-\frac{t^2}{2\tau^2} - \frac{x^2}{2R_x^2} - \frac{y^2}{2R_y^2} - \frac{z^2}{2R_z^2} \right]$$

$$S_r^{\text{eff}}(\mathbf{r}) = \frac{1}{8\pi^{3/2} \sqrt{R_x^2 + \mathrm{v}^2\tau^2} R_y R_z} \exp\left[-\frac{1}{4} \left(\frac{x^2}{R_x^2 + \mathrm{v}^2\tau^2} + \frac{y^2}{R_y^2} + \frac{z^2}{R_z^2} \right) \right]$$

$$R_s = R_x = R_y = R_z >> v\tau \qquad \boxed{S_r^{\text{eff}}(\mathbf{r}) = \frac{1}{8\pi^{3/2} R_s^3} \exp\left[-\frac{\mathbf{r}^2}{4R_s^2} \right]}$$

U. Heinz, B. Tomasik, U. A. Wiedemann and Y. F. Wu, Phys. Lett. B **382**, 181 (1996)

Source function in the CM frame

Source function transformed to the frame moving with respect to the source with **v** =(v,o,o)

$$S(L^{-1}x) = \frac{\sqrt{\det \Lambda'}}{4\pi^2} \exp\left[-\frac{1}{2} x'_\mu (L\Lambda L^{-1})^{\mu\nu} x'_\nu \right]$$

$$\det \Lambda' = \det \Lambda$$

$$\Lambda'^{\mu\nu} = \begin{bmatrix} \gamma^2(\tau^{-2} + v^2 R_x^{-2}) & -\gamma^2 v(\tau^{-2} + R_x^{-2}) & 0 & 0 \\ -\gamma^2 v(\tau^{-2} + R_x^{-2}) & \gamma^2(v^2\tau^{-2} + R_x^{-2}) & 0 & 0 \\ 0 & 0 & R_y^{-2} & 0 \\ 0 & 0 & 0 & R_z^{-2} \end{bmatrix}$$

Effective source function in the particle CM frame

$$\boxed{S_r^{\mathrm{eff}}(\mathbf{r}_*) = \frac{1}{8\pi^{3/2}\sqrt{\gamma^2(R_x^2 + \mathrm{v}^2\tau^2)}R_y R_z} \exp\left[-\frac{1}{4}\left(\frac{x_*^2}{\gamma^2(R_x^2 + \mathrm{v}^2\tau^2)} + \frac{y_*^2}{R_y^2} + \frac{z_*^2}{R_z^2} \right) \right]}$$

Correlation function

$$\boxed{C(\mathbf{P},\mathbf{q}) = \int d^3 r_* \, S_r^{\mathrm{eff}}(\mathbf{r}_*) \left|\varphi_{\mathbf{q}}(\mathbf{r}_*)\right|^2}$$

Effective source function in the source rest frame

$$S_r^{\mathrm{eff}}(\mathbf{r}) = \frac{1}{8\pi^{3/2} R_s^3} \exp\left[-\frac{\mathbf{r}^2}{4R_s^2}\right]$$ *isotropic*

Effective source function in the particle CM frame

$$S_r^{eff}(\mathbf{r}_*) = \frac{1}{8\pi^{3/2}\gamma R_s^3} \exp\left[-\frac{x_*^2}{4\gamma^2 R_s^2} - \frac{y_*^2 + z_*^2}{4R_s^2}\right]$$ *anisotropic*

The source is elongated by factor γ in the direction of particles' motion.

Correlation function

Lednicky-Lyuboshitz approach

$$\phi_{\mathbf{q}}(\mathbf{r}) = e^{i\mathbf{qr}} + f(\mathbf{q})\frac{e^{iqr}}{r}$$

wave function in scattering asymptotic state

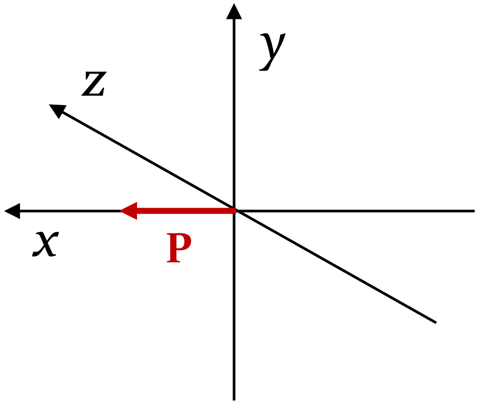
$$f(\mathbf{q}) = -\frac{a}{1-\frac{1}{2}d\,aq^2 + iqa}$$

s-wave amplidtue

a – scattering length, d – effective range

Correlation function of two spin 1/2 particles

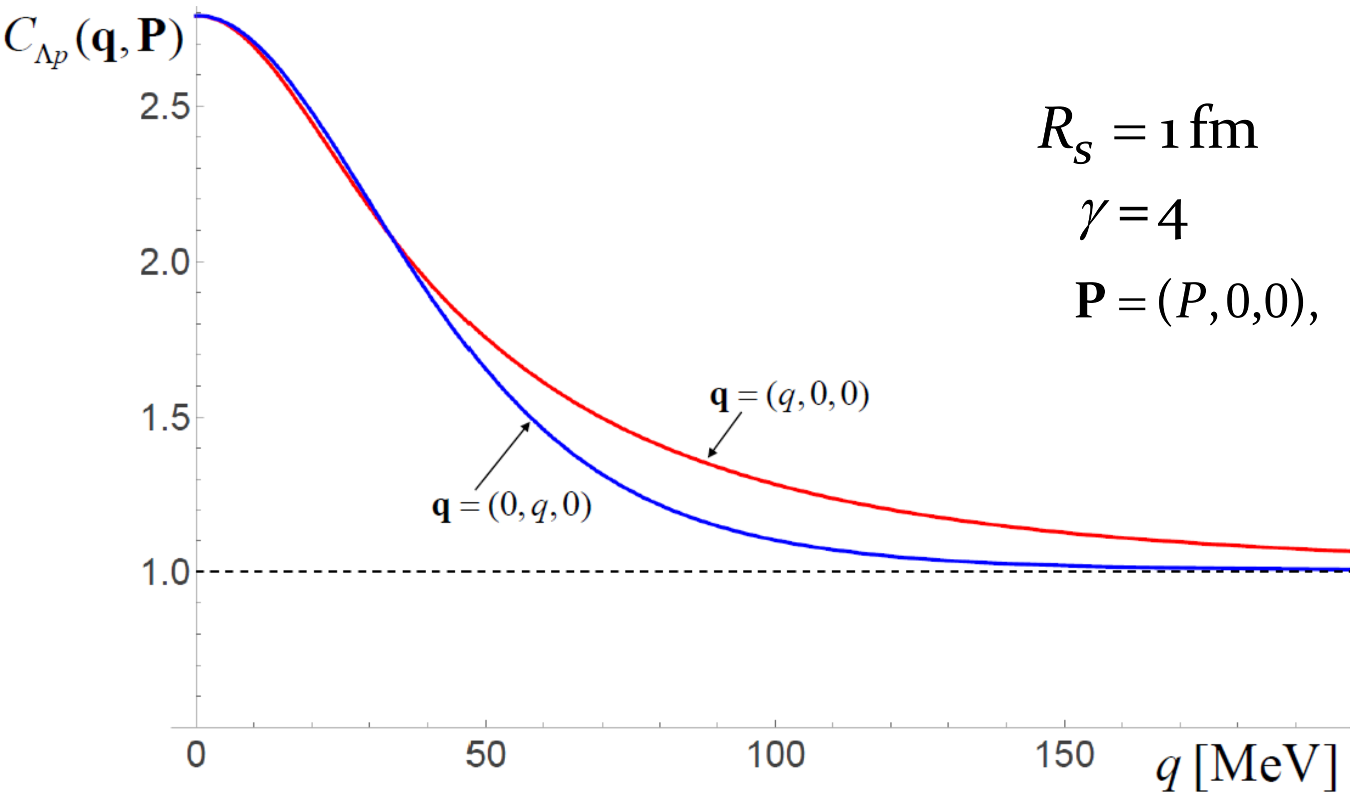
$$C(\mathbf{P},\mathbf{q}) = \frac{1}{4}C^s(\mathbf{P},\mathbf{q}) + \frac{1}{4}C^t(\mathbf{P},\mathbf{q})$$



$\mathbf{P} = (P,0,0)$, $\mathbf{q} = (q_x, q_y, q_z)$

R. Lednicky and V.L. Lyuboshitz, Yad. Fiz. **35**, 1316 (1982)

Λ-p correlation function



$R_s = 1$ fm

$\gamma = 4$

$\mathbf{P} = (P,0,0), \quad p_T \approx 3.6$ GeV

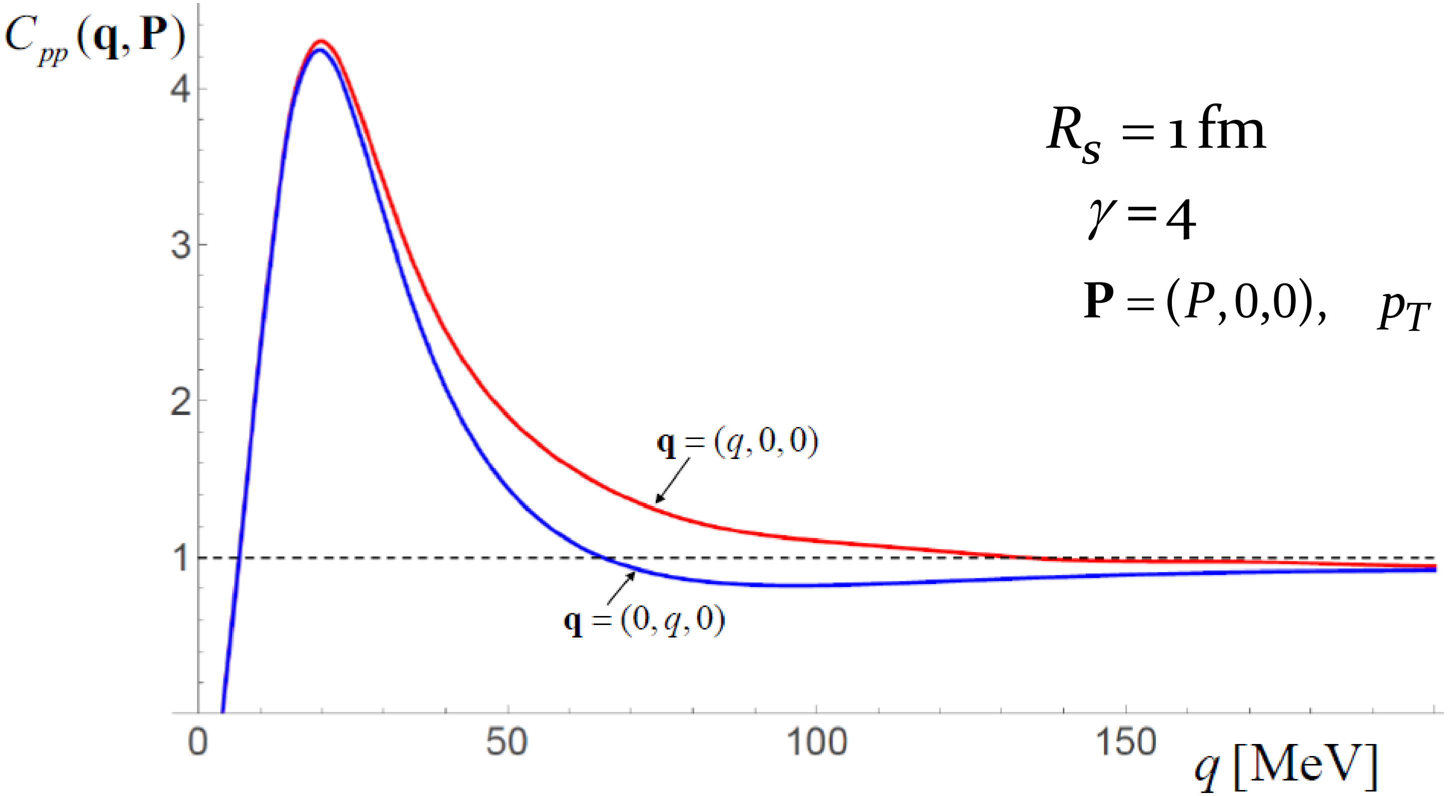
$a^s = -2.91$ fm $\quad a^t = -1.54$ fm

$d^s = 2.78$ fm $\quad d^t = 2.72$ fm

J. Haidenbauer et al. Nucl. Phys. A **915**, 24 (2013).

St. Mrówczyński, arXiv:2603.00354 to appear in Phys. Rev. C

p-p correlation function



$R_s = 1$ fm

$\gamma = 4$

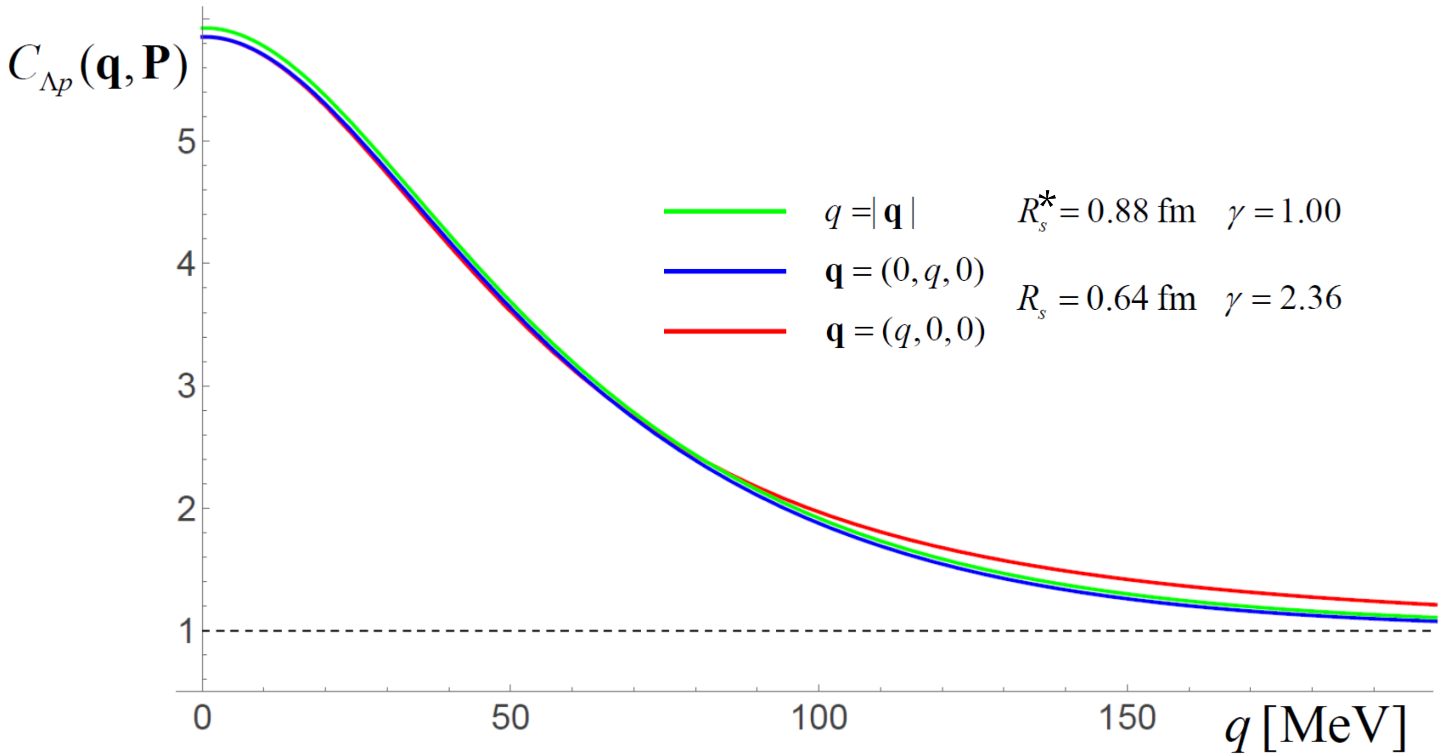
$\mathbf{P} = (P,0,0), \quad p_T \approx 3.6$ GeV

$a^s = -7.8$ fm

$d^s = 2.8$ fm

St. Mrówczyński, arXiv:2603.00354 to appear in Phys. Rev. C

Are relativistic effects observable?



$$\gamma = 2.36 \quad \Rightarrow \quad p_T \approx 2 \text{ GeV}$$

St. Mrówczyński, arXiv:2603.00354 to appear in Phys. Rev. C

Deuteron formation

Deuteron yield

isospin factor

$$\frac{dN^D}{d^3\mathbf{P}_D} = \frac{1}{2} A_D \frac{dN^{np}}{d^3\mathbf{p}_n d^3\mathbf{p}_p} \qquad \frac{1}{2}\mathbf{P}_D = \mathbf{p}_n = \mathbf{p}_p$$

deuteron formation rate

yield of *np* pairs

$$\frac{1}{2}\frac{dN^{np}}{d^3\mathbf{p}_n d^3\mathbf{p}_p} \approx \frac{dN^{pp}}{d^3\mathbf{p}_p d^3\mathbf{p}_p} \approx \left(\frac{dN^p}{d^3\mathbf{p}_p}\right)^2$$

$$\boxed{\frac{dN^D}{d^3\mathbf{P}_D} = A_D \left(\frac{dN^p}{d^3\mathbf{p}_p}\right)^2}$$

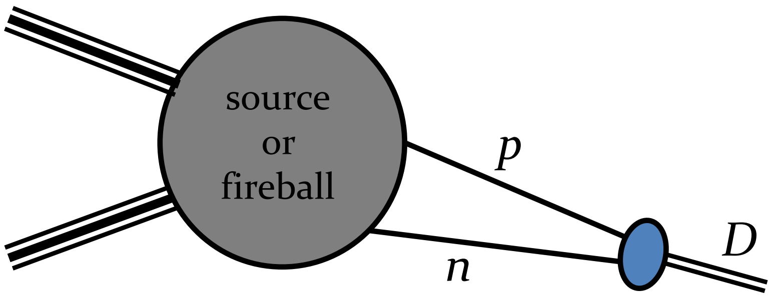
$$\boxed{E_D \frac{dN^D}{d^3\mathbf{P}_D} = B \left(E_p \frac{dN^p}{d^3\mathbf{p}_p}\right)^2}$$

Lorentz invariant coalsecence factor

$$B = \frac{2}{E_p} A_D = \frac{2}{m\gamma} A_D$$

S.T. Butler & C.A. Pearson, Phys. Rev. **129**, 836 (1963)
A. Schwarzschild & C. Zupancic, Phys. Rev. **129**, 854 (1963)

Sato-Yazaki formula



$$A_D = \frac{3}{4}(2\pi)^3 \int dt_p d^3 r_p \int dt_n d^3 r_n \, S^p(\mathbf{r}_p) S^n(\mathbf{r}_n) \left|\psi(\mathbf{r}_p + \mathbf{v}t_p, \mathbf{r}_n + \mathbf{v}t_n)\right|^2$$

spin factor

•
• as in case of correlation function
•

$$\boxed{A_D = \frac{3}{4}(2\pi)^3 \int d^3 r \, S_r^{\text{eff}}(\mathbf{r}) \left|\varphi_D(\mathbf{r})\right|^2}$$

$$B = \frac{2}{m\gamma} A_D^*$$

γ - gamma factor of the transformation to the *n*-*p* CM frame

H. Sato and K. Yazaki, Phys. Lett. B **98**, 153 (1981)

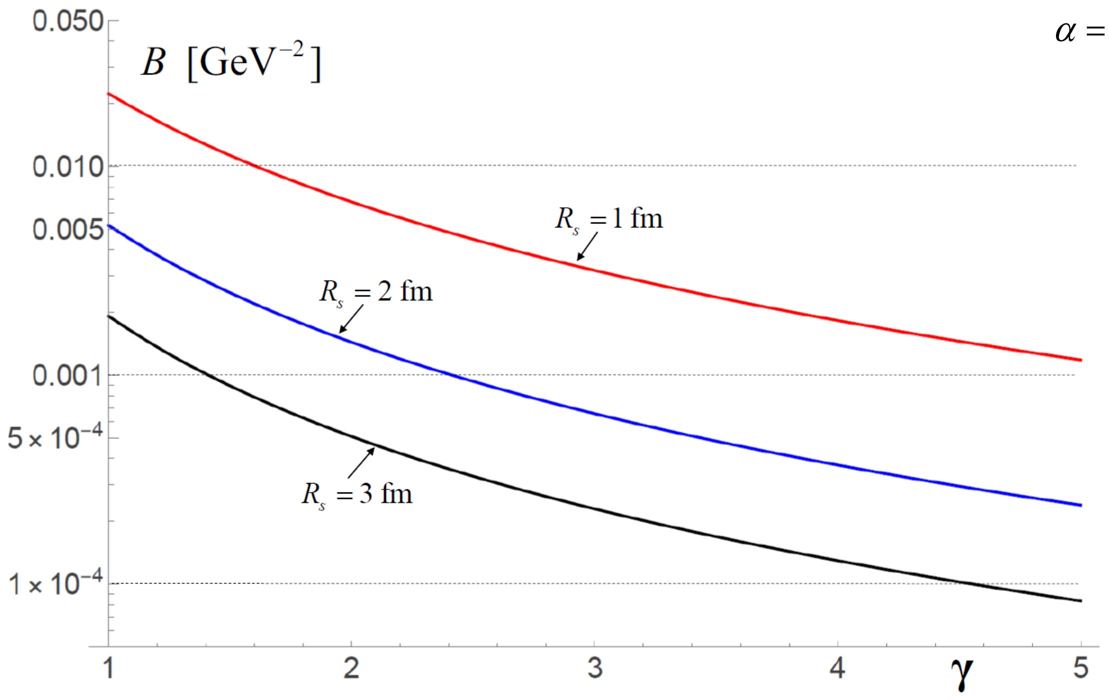
Coalescence coefficient

$$B = \frac{3}{2m\gamma}(2\pi)^3 \int d^3 r_* \, S_r^{\mathrm{eff}}(\mathbf{r}_*) \left|\varphi_D(\mathbf{r}_*)\right|^2$$

Hulthén wave function

$$\varphi_D(r) = \sqrt{\frac{\alpha\beta(\alpha+\beta)}{2\pi(\alpha-\beta)^2}} \, \frac{\exp(-\alpha r) - \exp(-\beta r)}{r}$$

$$\alpha = 0.23 \, \mathrm{fm}^{-1}, \quad \beta = 1.61 \, \mathrm{fm}^{-1}$$



St. Mrówczyński, arXiv:2603.00354 to appear in Phys. Rev. C

Deuteron production in p-p @ LHC

Experimental value

$$B = (1.2 \pm 0.2) \cdot 10^{-2}\,\mathrm{GeV}^2 \;@\; p_T = 2\,\mathrm{GeV}$$

S. Acharya et al. [ALICE], JHEP **01**, 106 (2022)

Theoretical value

$$p_T = 2\,\mathrm{GeV} \;\Rightarrow\; R_s^* = 0.88\,\mathrm{fm} \;\Rightarrow\; B = 2.9 \cdot 10^{-2}\,\mathrm{GeV}^2 \,?$$

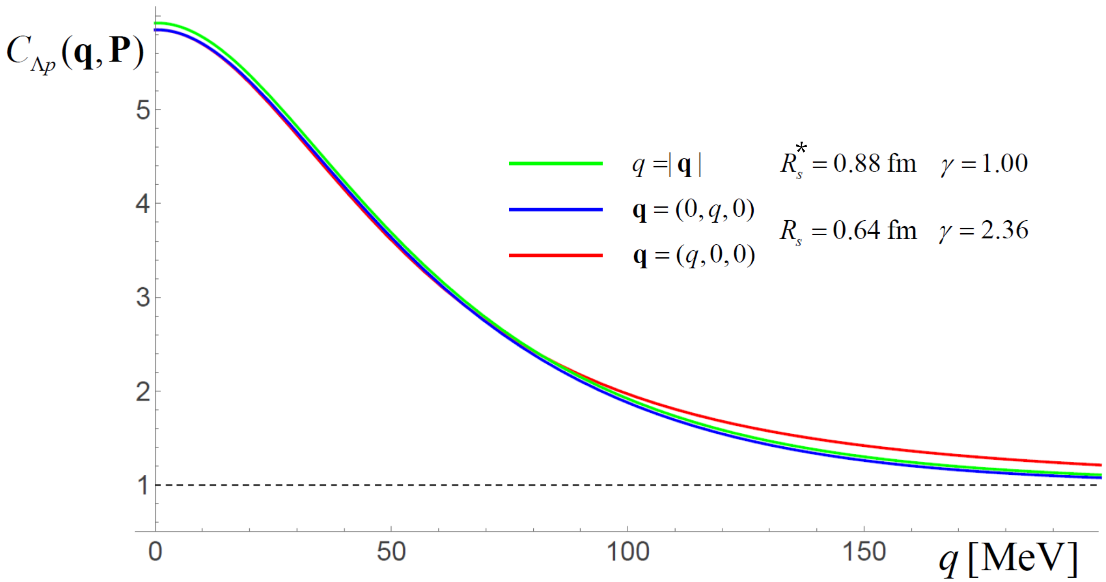
S. Acharya et al. [ALICE], Phys. Lett. B **811**, 135849 (2020); erratum: B 861, 139233 (2025)

Relativistic effects are not taken into account!

$$p_T = 2\,\mathrm{GeV} \;\Rightarrow\; \gamma = 2.36$$

St. Mrówczyński, arXiv:2603.00354 to appear in Phys. Rev. C

Source radius from Λ-p correlations



$$B = 2.9 \cdot 10^{-2}\,\mathrm{GeV}^2 \quad \text{for} \quad \gamma = 1 \quad \& \quad R_s^* = 0.88\,\mathrm{fm}$$

$$B = 1.1 \cdot 10^{-2}\,\mathrm{GeV}^2 \quad \text{for} \quad \gamma = 2.36 \quad \& \quad R_s = 0.64\,\mathrm{fm}$$

Experimental value

$$B = (1.2 \pm 0.2) \cdot 10^{-2}\,\mathrm{GeV}^2 \quad @ \quad p_T = 2\,\mathrm{GeV}$$

St. Mrówczyński, arXiv:2603.00354 to appear in Phys. Rev. C

Conclusions

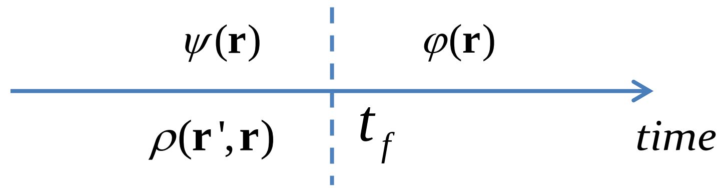
- A particle source, which isotropic in its rest frame, is anisotropic in the particles' CM frame.
- Femtoscopic correlation functions weakly depend on relativistic effects.
- Coalescence coefficients of light nuclei are sensitive to relativistic effects.
- Combined analysis of femtoscopic correlations and formation of light nuclei allows one to pin down relativistic effects.

Back-up

Quantum-mechanical meaning of the formation rate formula

Sudden approximation

$$E\Delta t << 1$$



Transition matrix element

$$M = \left| \int d^3\mathbf{r}\, \psi^*(\mathbf{r})\, \varphi(\mathbf{r}) \right|^2 = \int d^3\mathbf{r}\, d^3\mathbf{r}'\, \varphi^*(\mathbf{r}')\, \underbrace{\psi(\mathbf{r}')\psi^*(\mathbf{r})}_{\rho(\mathbf{r}',\mathbf{r})\ \text{density matrix}}\, \varphi(\mathbf{r})$$

$$M = \int d^3\mathbf{r}\, d^3\mathbf{r}'\, \varphi^*(\mathbf{r}')\, \rho(\mathbf{r}',\mathbf{r})\, \varphi(\mathbf{r})$$

If density matrix is diagonal

$$\rho(\mathbf{r}',\mathbf{r}) = S(\mathbf{r})\, \delta^{(3)}(\mathbf{r}'-\mathbf{r}) \qquad \Rightarrow \qquad \boxed{M = \int d^3\mathbf{r}\, S(\mathbf{r}) \left| \varphi(\mathbf{r}) \right|^2}$$

Diagonal density matrix

$$\langle\psi|\hat{A}|\psi\rangle = \sum_{i,j} c_i^* c_j \langle\alpha_i|\hat{A}|\alpha_j\rangle = \sum_{i,j} \rho_{ji} A_{ij}$$

$$|\psi\rangle = \sum_i c_i |\alpha_i\rangle$$

$$\rho_{ji} \equiv c_i^* c_j$$

$$A_{ij} \equiv \langle\alpha_i|\hat{A}|\alpha_j\rangle$$

density matrix

$\overline{\cdots\cdots}$ – averaging over time or events

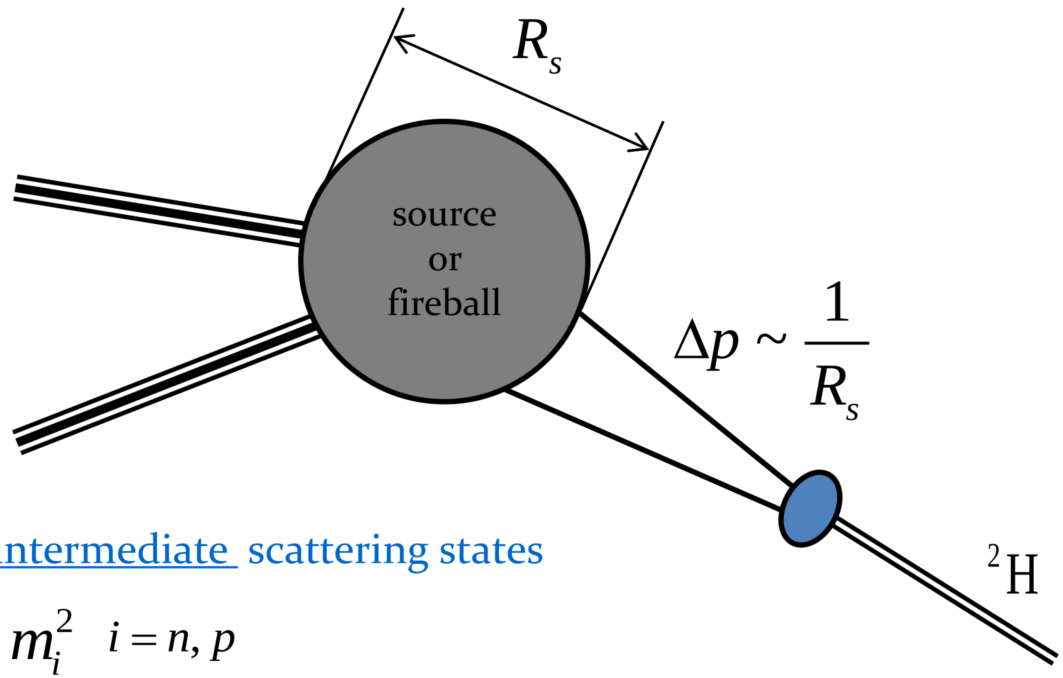
$$\overline{\langle\psi|\hat{A}|\psi\rangle} = \sum_{i,j} \overline{c_i^* c_j} \langle\alpha_i|\hat{A}|\alpha_j\rangle = \sum_i |c_i|^2 A_{ii}$$

$$\overline{\rho_{ji}} = \overline{c_i^* c_j} = \delta^{ij} |c_i|^2$$

random phase approximation

diagonal density matrix

Energy-momentum conservation



Nucleons are intermediate scattering states

$$E_i^2 - \mathbf{p}_i^2 \neq m_i^2 \quad i = n, p$$

Energy-momentum conservation

$$\begin{cases} \mathbf{p}_p + \mathbf{p}_n = \mathbf{p}_D \\ E_p + E_n = E_D \end{cases}$$

St. Mrówczyński, J. Phys. G **11**, 1087 (1987)