

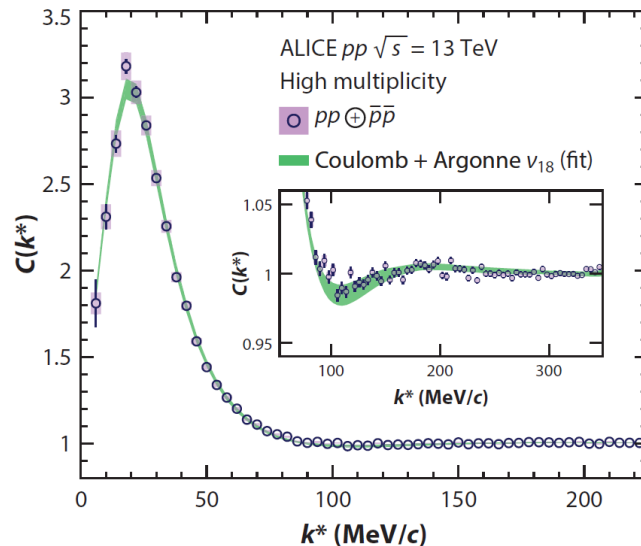
Relativistic Effects in Femtoscopy & Deuteron Formation

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Why relativistic effects need to be discussed?

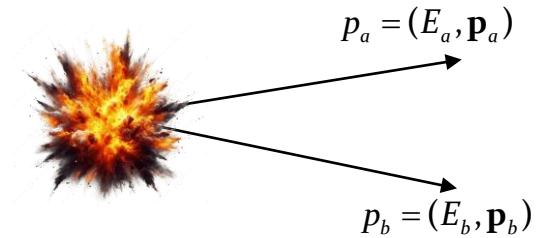
- ▶ Measurements of two particle correlations are usually performed in the CM frame where the particles are non-relativistic. The motion of the pair with respect to the particles' source is truly relativistic.
- ▶ Femtoscopic measurements are very precise nowadays. Are theoretical predictions equally accurate?



Correlation function

Definition

$$\frac{dN_{ab}}{d\mathbf{p}_a d\mathbf{p}_b} = C(\mathbf{p}_a, \mathbf{p}_b) \frac{dN_a}{d\mathbf{p}_a} \frac{dN_b}{d\mathbf{p}_b}$$



Correlation function
is a Lorentz scalar

$$\underbrace{E_a E_b \frac{dN_{ab}}{d\mathbf{p}_a d\mathbf{p}_b}}_{\text{Lorentz invariant}} = C(p_a, p_b) \underbrace{E_a \frac{dN_a}{d\mathbf{p}_a} E_b \frac{dN_b}{d\mathbf{p}_b}}_{\text{Lorentz invariant}}$$

Lorentz transformation $p \equiv (E, \mathbf{p}) \rightarrow p' \equiv (E', \mathbf{p}') = Lp$

$$C(p_a, p_b) \rightarrow C'(p'_a, p'_b) = C(L^{-1} p'_a, L^{-1} p'_b)$$

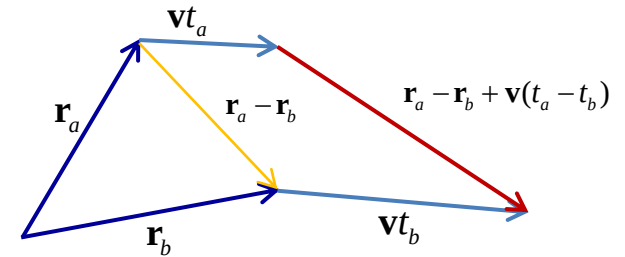
Non-relativistic Koonin formula

$$C(\mathbf{p}_a, \mathbf{p}_b) = \int dt_a d^3 r_a \int dt_b d^3 r_b S^a(t_a, \mathbf{r}_a) S^b(t_b, \mathbf{r}_b) \left| \psi(\mathbf{r}_a + \mathbf{v}t_a, \mathbf{r}_b + \mathbf{v}t_b) \right|^2$$

wave function

source functions

$$\int dt d^3 \mathbf{r} S^{a,b}(t, \mathbf{r}) = 1$$



CM variables

$$\left\{ \begin{array}{ll} \mathbf{R} \equiv \frac{m_a \mathbf{r}_a + m_b \mathbf{r}_b}{m_a + m_b}, & \mathbf{r} \equiv \mathbf{r}_a - \mathbf{r}_b \\ T \equiv \frac{m_a t_a + m_b t_b}{m_a + m_b}, & t \equiv t_a - t_b \\ \mathbf{P} \equiv \mathbf{p}_a + \mathbf{p}_b, & \mathbf{q} \equiv \frac{m_b \mathbf{p}_a + m_a \mathbf{p}_b}{m_a + m_b} \end{array} \right.$$

$$S_r(t, \mathbf{r}) \equiv \int dT d^3 \mathbf{R} S^a\left(T - \frac{1}{2}t, \mathbf{R} - \frac{1}{2}\mathbf{r}\right) S^b\left(T + \frac{1}{2}t, \mathbf{R} + \frac{1}{2}\mathbf{r}\right) \quad \psi(\mathbf{r}_a, \mathbf{r}_b) = e^{i\mathbf{P} \cdot \mathbf{R}} \varphi_{\mathbf{q}}(\mathbf{r})$$

$$S_r^{\text{eff}}(\mathbf{r}) \equiv \int dt S_r(t, \mathbf{r} - \mathbf{v}t)$$

$$\int d^3 \mathbf{r} S_r^{\text{eff}}(\mathbf{r}) = 1$$

$$C(\mathbf{P}, \mathbf{q}) = \int d^3 r S_r^{\text{eff}}(\mathbf{r}) \left| \varphi_{\mathbf{q}}(\mathbf{r}) \right|^2$$

The formula holds in the CM frame
of correlated particles!

Relativistically covariant source function

Emission probability is Lorentz invariant $S(x)d^4x = S'(x')d^4x' \quad x \equiv (t, \mathbf{r})$

Since $d^4x = d^4x'$ the source function is a Lorentz scalar $S(x) \rightarrow S'(x') = S(L^{-1}x')$

Gaussian source function

$$S(x) = \frac{\sqrt{\det \Lambda}}{4\pi^2} \exp \left[-\frac{1}{2} x_\mu \Lambda^{\mu\nu} x_\nu \right]$$

In source rest frame

$$\Lambda^{\mu\nu} = \begin{bmatrix} \tau^{-2} & 0 & 0 & 0 \\ 0 & R_x^{-2} & 0 & 0 \\ 0 & 0 & R_y^{-2} & 0 \\ 0 & 0 & 0 & R_z^{-2} \end{bmatrix}$$

$$S(x) = \frac{1}{4\pi^2 \tau R_x R_y R_z} \exp \left[-\frac{t^2}{2\tau^2} - \frac{x^2}{2R_x^2} - \frac{y^2}{2R_y^2} - \frac{z^2}{2R_z^2} \right]$$

$$S_r^{\text{eff}}(\mathbf{r}) = \frac{1}{8\pi^{3/2} \sqrt{R_x^2 + v^2 \tau^2} R_y R_z} \exp \left[-\frac{1}{4} \left(\frac{x^2}{R_x^2 + v^2 \tau^2} + \frac{y^2}{R_y^2} + \frac{z^2}{R_z^2} \right) \right]$$

$$R_s = R_x = R_y = R_z \gg v\tau$$

$$S_r^{\text{eff}}(\mathbf{r}) = \frac{1}{8\pi^{3/2} R_s^3} \exp \left[-\frac{\mathbf{r}^2}{4R_s^2} \right]$$

Source function in the CM frame

Source function transformed to the frame moving with respect to the source with $\mathbf{v}=(v,o,o)$

$$S(L^{-1}x) = \frac{\sqrt{\det \Lambda'}}{4\pi^2} \exp \left[-\frac{1}{2} x'_\mu (L\Lambda L^{-1})^{\mu\nu} x'_\nu \right]$$

$$\det \Lambda' = \det \Lambda$$

$$\Lambda'^{\mu\nu} = \begin{bmatrix} \gamma^2(\tau^{-2} + v^2 R_x^{-2}) & -\gamma^2 v(\tau^{-2} + R_x^{-2}) & 0 & 0 \\ -\gamma^2 v(\tau^{-2} + R_x^{-2}) & \gamma^2(v^2 \tau^{-2} + R_x^{-2}) & 0 & 0 \\ 0 & 0 & R_y^{-2} & 0 \\ 0 & 0 & 0 & R_z^{-2} \end{bmatrix}$$

Effective source function in the particle CM frame

$$S_r^{\text{eff}}(\mathbf{r}_*) = \frac{1}{8\pi^{3/2} \sqrt{\gamma^2(R_x^2 + v^2 \tau^2)} R_y R_z} \exp \left[-\frac{1}{4} \left(\frac{x_*^2}{\gamma^2(R_x^2 + v^2 \tau^2)} + \frac{y_*^2}{R_y^2} + \frac{z_*^2}{R_z^2} \right) \right]$$

Correlation function

$$C(\mathbf{P}, \mathbf{q}) = \int d^3r_* S_r^{\text{eff}}(\mathbf{r}_*) |\varphi_{\mathbf{q}}(\mathbf{r}_*)|^2$$

Effective source function in the source rest frame

$$S_r^{\text{eff}}(\mathbf{r}) = \frac{1}{8\pi^{3/2} R_s^3} \exp\left[-\frac{\mathbf{r}^2}{4R_s^2}\right] \quad \textit{isotropic}$$

Effective source function in the particle CM frame

$$S_r^{\text{eff}}(\mathbf{r}_*) = \frac{1}{8\pi^{3/2} \gamma R_s^3} \exp\left[-\frac{x_*^2}{4\gamma^2 R_s^2} - \frac{y_*^2 + z_*^2}{4R_s^2}\right] \quad \textit{anisotropic}$$

The source is elongated by factor γ in the direction of particles' motion.

Correlation function

Lednickiy-Lyuboshitz approach

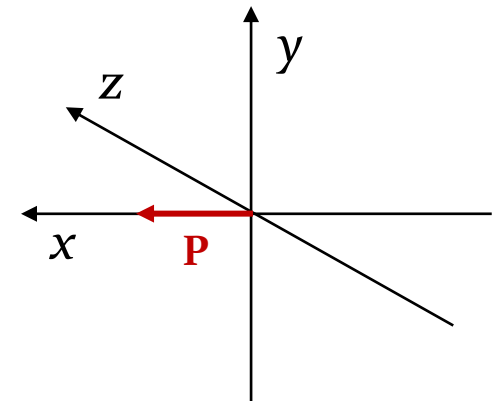
$$\phi_{\mathbf{q}}(\mathbf{r}) = e^{i\mathbf{q}\mathbf{r}} + f(\mathbf{q}) \frac{e^{iqr}}{r} \quad \text{wave function in scattering asymptotic state}$$

$$f(\mathbf{q}) = -\frac{a}{1 - \frac{1}{2}daq^2 + iqa} \quad \text{s-wave amplitude}$$

a – scattering length, d – effective range

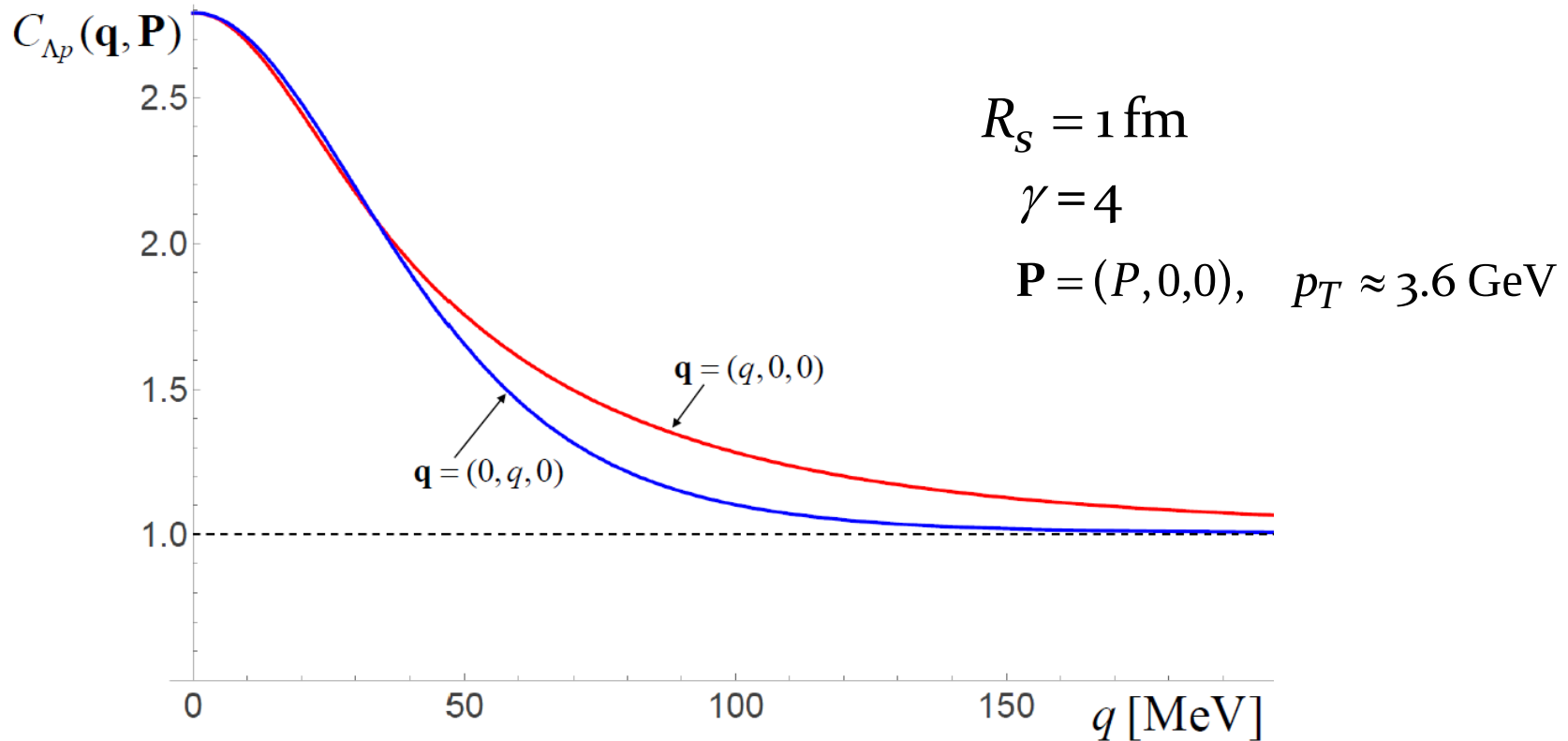
Correlation function of two spin 1/2 particles

$$C(\mathbf{P}, \mathbf{q}) = \frac{1}{4} C^s(\mathbf{P}, \mathbf{q}) + \frac{1}{4} C^t(\mathbf{P}, \mathbf{q})$$



$$\mathbf{P} = (P, 0, 0), \quad \mathbf{q} = (q_x, q_y, q_z)$$

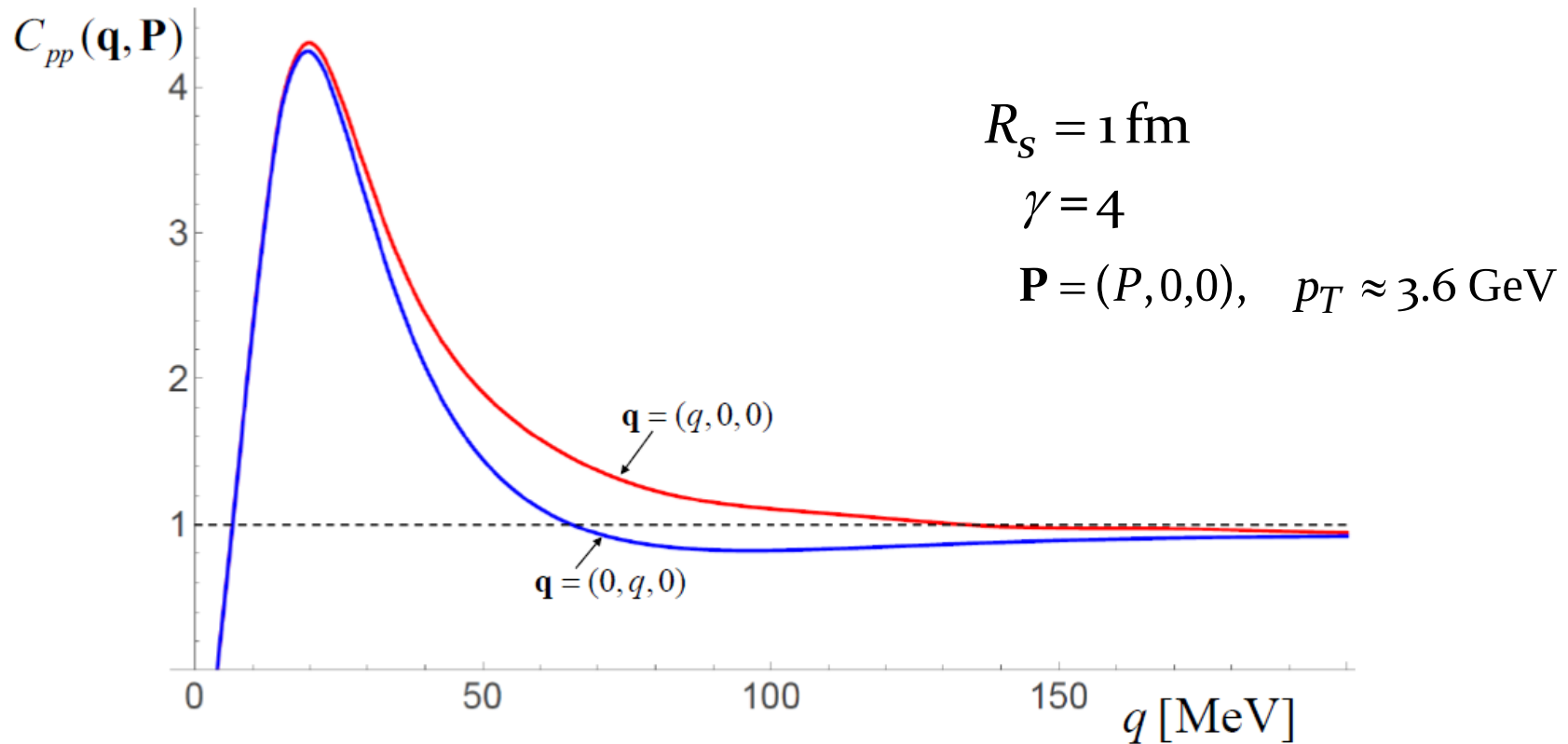
Λ -p correlation function



$$\begin{aligned}
 a^s &= -2.91 \text{ fm} & a^t &= -1.54 \text{ fm} \\
 d^s &= 2.78 \text{ fm} & d^t &= 2.72 \text{ fm}
 \end{aligned}$$

J. Haidenbauer et al. Nucl. Phys. A **915**, 24 (2013).

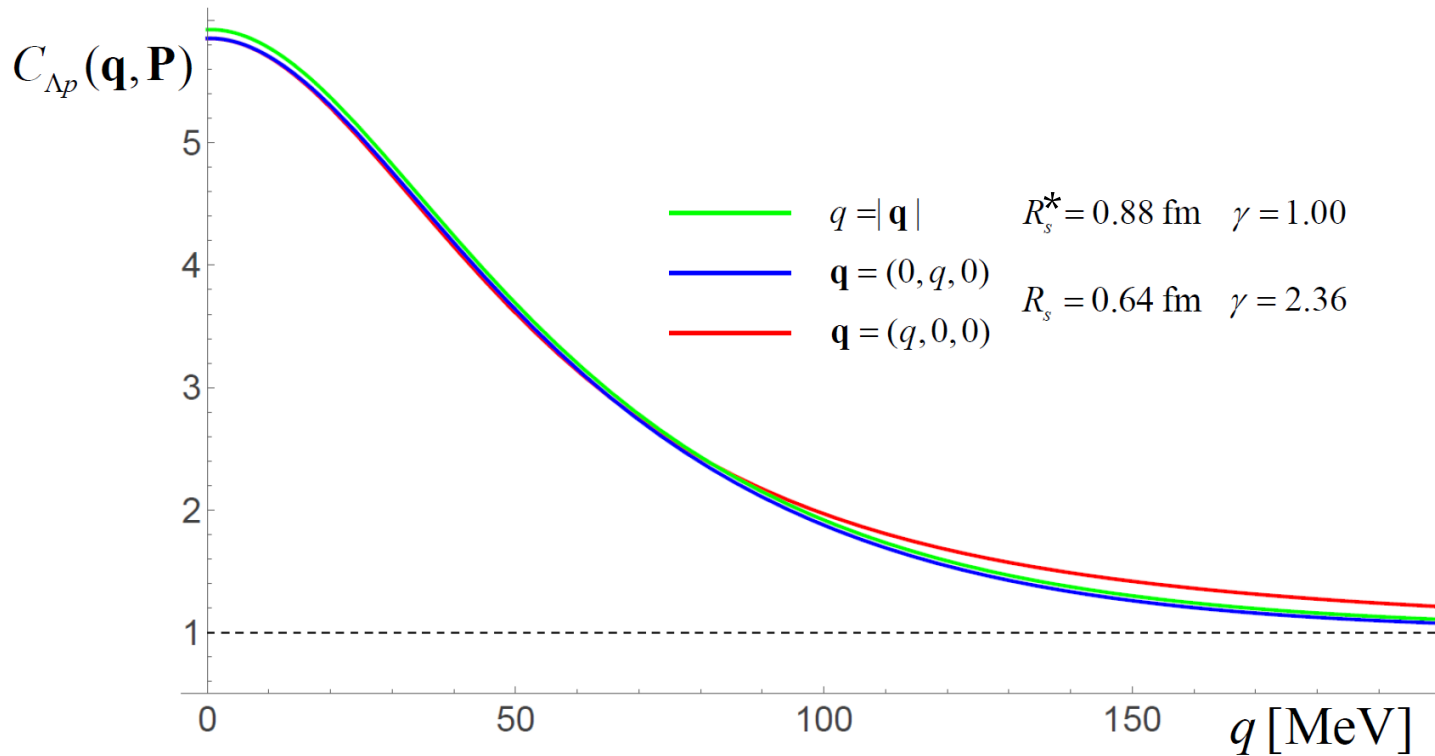
p-p correlation function



$$a^S = -7.8 \text{ fm}$$

$$d^S = 2.8 \text{ fm}$$

Are relativistic effects observable?



$$\gamma = 2.36 \Rightarrow p_T \approx 2 \text{ GeV}$$

Deuteron formation

Deuteron yield

$$\boxed{\frac{dN^D}{d^3\mathbf{P}_D} = \frac{1}{2} A_D \frac{dN^{np}}{d^3\mathbf{p}_n d^3\mathbf{p}_p}}$$

isospin factor
deuteron formation rate
yield of np pairs

$$\frac{1}{2} \mathbf{P}_D = \mathbf{p}_n = \mathbf{p}_p$$

$$\frac{1}{2} \frac{dN^{np}}{d^3\mathbf{p}_n d^3\mathbf{p}_p} \approx \frac{dN^{pp}}{d^3\mathbf{p}_p d^3\mathbf{p}_p} \approx \left(\frac{dN^p}{d^3\mathbf{p}_p} \right)^2$$

$$\boxed{\frac{dN^D}{d^3\mathbf{P}_D} = A_D \left(\frac{dN^p}{d^3\mathbf{p}_p} \right)^2}$$

$$\boxed{E_D \frac{dN^D}{d^3\mathbf{P}_D} = B \left(E_p \frac{dN^p}{d^3\mathbf{p}_p} \right)^2}$$

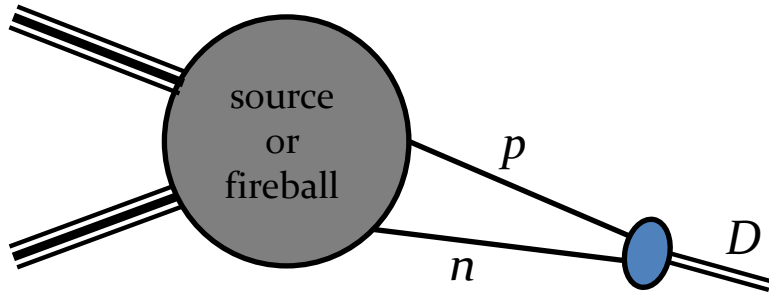
Lorentz invariant coalescence factor

$$B = \frac{2}{E_p} A_D = \frac{2}{m\gamma} A_D$$

S.T. Butler & C.A. Pearson, Phys. Rev. **129**, 836 (1963)

A. Schwarzschild & C. Zupancic, Phys. Rev. **129**, 854 (1963)

Sato-Yazaki formula



$$A_D = \frac{3}{4} (2\pi)^3 \int dt_p d^3 r_p \int dt_n d^3 r_n S^p(t_p, \mathbf{r}_p) S^n(t_n, \mathbf{r}_n) \left| \psi(\mathbf{r}_p + \mathbf{v}t_p, \mathbf{r}_n + \mathbf{v}t_n) \right|^2$$

spin factor

•
• as in case of correlation function
•

$$A_D = \frac{3}{4} (2\pi)^3 \int d^3 r S_r^{\text{eff}}(\mathbf{r}) \left| \varphi_D(\mathbf{r}) \right|^2$$

$$B = \frac{2}{m\gamma} A_D^*$$

γ - gamma factor of the transformation to the n - p CM frame

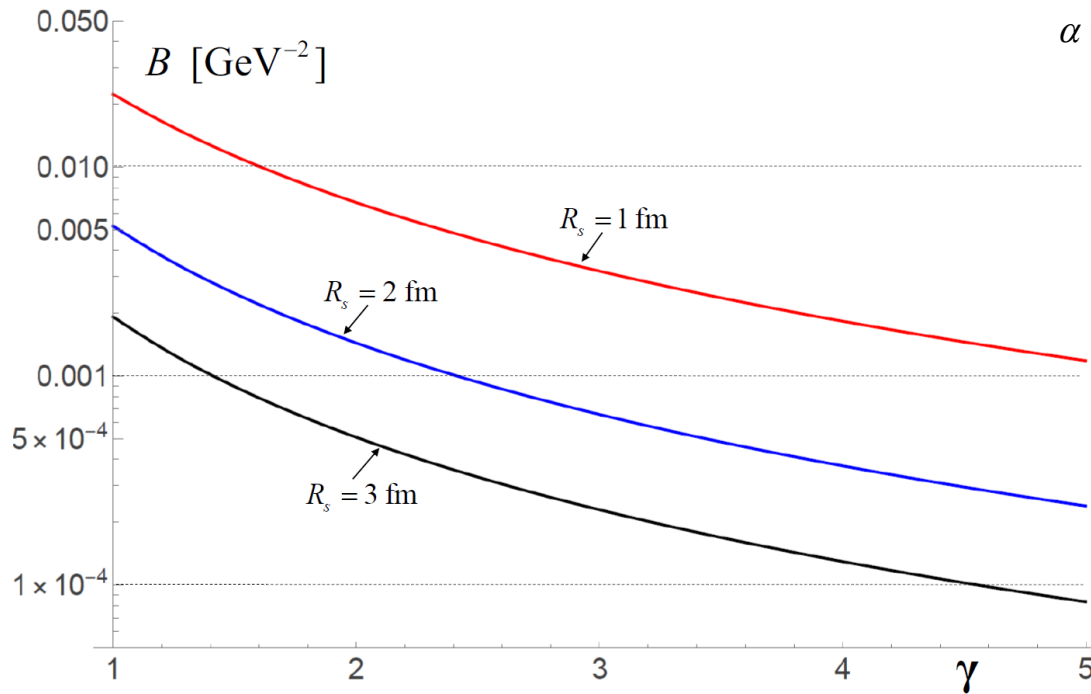
Coalescence coefficient

$$B = \frac{3}{2m\gamma} (2\pi)^3 \int d^3r_* S_r^{\text{eff}}(\mathbf{r}_*) |\varphi_D(\mathbf{r}_*)|^2$$

Hulthén wave function

$$\varphi_D(r) = \sqrt{\frac{\alpha\beta(\alpha+\beta)}{2\pi(\alpha-\beta)^2}} \frac{\exp(-\alpha r) - \exp(-\beta r)}{r}$$

$$\alpha = 0.23 \text{ fm}^{-1}, \quad \beta = 1.61 \text{ fm}^{-1}$$



Deuteron production in p-p @ LHC

Experimental value

$$\underline{B = (1.2 \pm 0.2) \cdot 10^{-2} \text{ GeV}^2} @ p_T = 2 \text{ GeV}$$

S. Acharya et al. [ALICE], JHEP **01**, 106 (2022)

Theoretical value

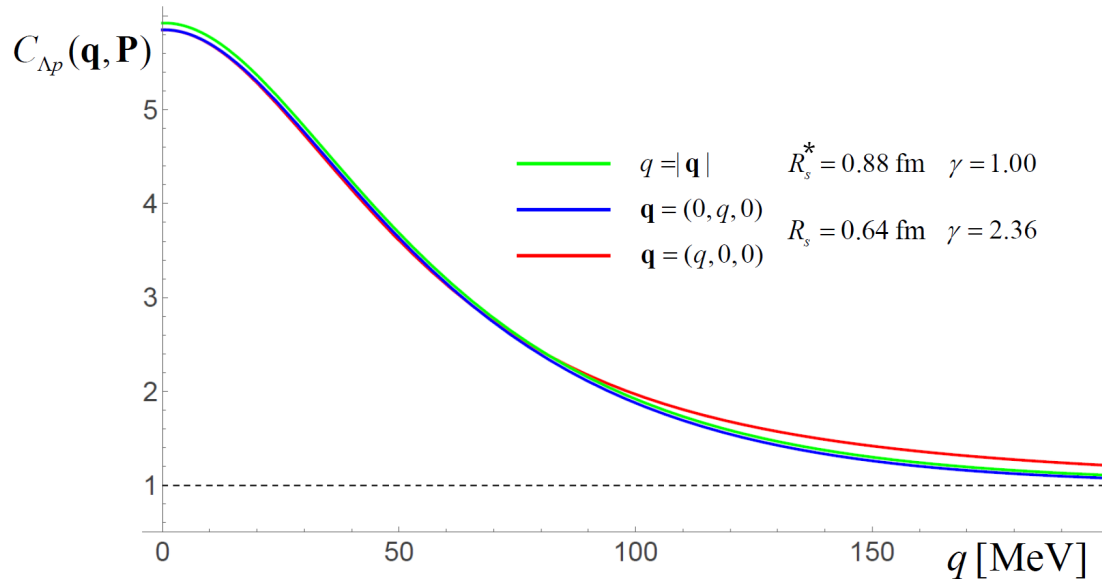
$$p_T = 2 \text{ GeV} \Rightarrow R_s^* = 0.88 \text{ fm} \Rightarrow \underline{B = 2.9 \cdot 10^{-2} \text{ GeV}^2} ?$$

S. Acharya et al. [ALICE], Phys. Lett. B **811**, 135849 (2020); erratum: B **861**, 139233 (2025)

Relativistic effects are not taken into account!

$$p_T = 2 \text{ GeV} \Rightarrow \gamma = 2.36$$

Source radius from Λ -p correlations



$$B = 2.9 \cdot 10^{-2} \text{ GeV}^2 \text{ for } \gamma = 1 \quad \& \quad R_s^* = 0.88 \text{ fm}$$

$$\underline{B = 1.1 \cdot 10^{-2} \text{ GeV}^2 \text{ for } \gamma = 2.36 \quad \& \quad R_s = 0.64 \text{ fm}}$$

Experimental value

$$\underline{B = (1.2 \pm 0.2) \cdot 10^{-2} \text{ GeV}^2 \text{ @ } p_T = 2 \text{ GeV}}$$

Conclusions

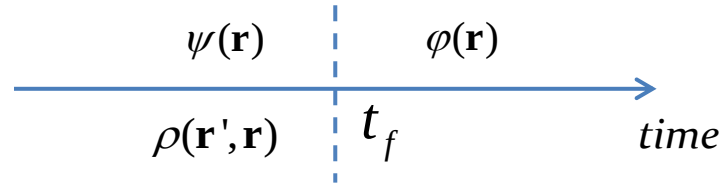
- ▶ A particle source, which is isotropic in its rest frame, is anisotropic in the particles' CM frame.
- ▶ Femtoscopic correlation functions weakly depend on relativistic effects.
- ▶ Coalescence coefficients of light nuclei are sensitive to relativistic effects.
- ▶ Combined analysis of femtoscopic correlations and formation of light nuclei allows one to pin down relativistic effects.

Back-up

Quantum-mechanical meaning of the formation rate formula

Sudden approximation

$$E \Delta t \ll 1$$



Transition matrix element

$$M = \left| \int d^3\mathbf{r} \psi^*(\mathbf{r}) \phi(\mathbf{r}) \right|^2 = \int d^3\mathbf{r} d^3\mathbf{r}' \phi^*(\mathbf{r}') \underbrace{\psi(\mathbf{r}') \psi^*(\mathbf{r})}_{\substack{\text{density matrix} \\ \rho(\mathbf{r}', \mathbf{r})}} \phi(\mathbf{r})$$

$$M = \int d^3\mathbf{r} d^3\mathbf{r}' \phi^*(\mathbf{r}') \rho(\mathbf{r}', \mathbf{r}) \phi(\mathbf{r})$$

If density matrix is diagonal

$$\rho(\mathbf{r}', \mathbf{r}) = S(\mathbf{r}) \delta^{(3)}(\mathbf{r}' - \mathbf{r}) \quad \Rightarrow$$

$$M = \int d^3\mathbf{r} S(\mathbf{r}) |\phi(\mathbf{r})|^2$$

Diagonal density matrix

$$\langle \psi | \hat{A} | \psi \rangle = \sum_{i,j} c_i^* c_j \langle \alpha_i | \hat{A} | \alpha_j \rangle = \sum_{i,j} \rho_{ji} A_{ij}$$

$$|\psi\rangle = \sum_i c_i |\alpha_i\rangle$$

$$\rho_{ji} \equiv c_i^* c_j$$

$$A_{ij} \equiv \langle \alpha_i | \hat{A} | \alpha_j \rangle$$

density matrix

..... – averaging over time or events

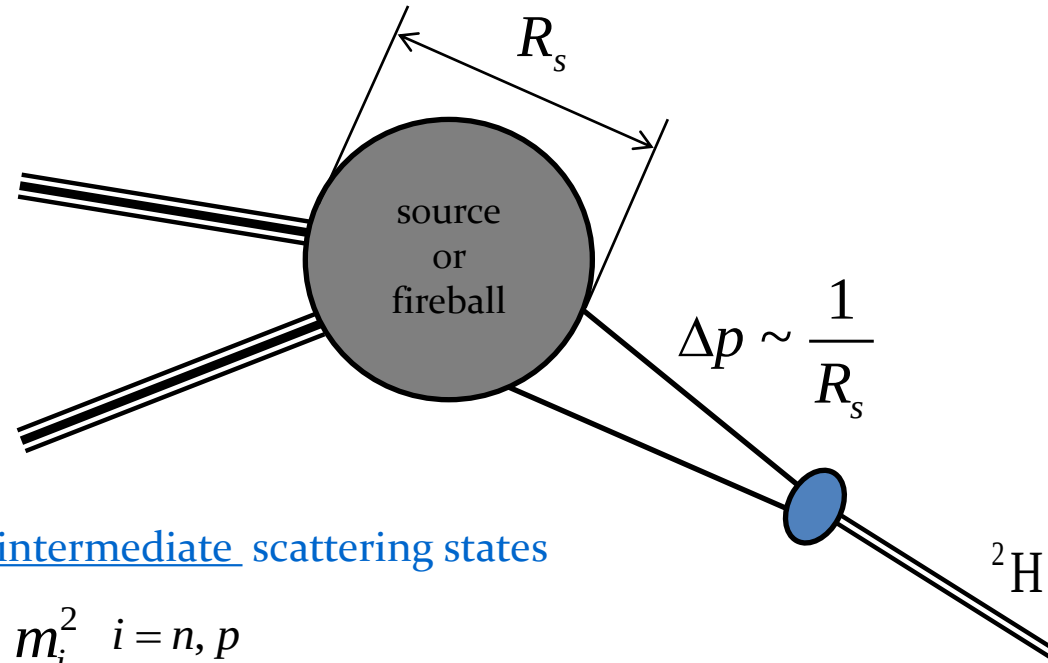
$$\overline{\langle \psi | \hat{A} | \psi \rangle} = \sum_{i,j} \overline{c_i^* c_j} \langle \alpha_i | \hat{A} | \alpha_j \rangle = \sum_i |c_i|^2 A_{ii}$$

$$\overline{\rho_{ji}} = \overline{c_i^* c_j} = \delta^{ij} |c_i|^2$$

random phase approximation

diagonal density matrix

Energy-momentum conservation



Nucleons are intermediate scattering states

$$E_i^2 - \mathbf{p}_i^2 \neq m_i^2 \quad i = n, p$$

Energy-momentum conservation

$$\begin{cases} \mathbf{p}_p + \mathbf{p}_n = \mathbf{p}_D \\ E_p + E_n = E_D \end{cases}$$